FORMATION OF WAVES OF FINITE AMPLITUDE BY A FLUID SOURCE

(OBRAZOVANIE VOLN KONECHNOI AMPLITUDY ISTOCHNIKOM ZHIDKOSTI)

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1. Let us consider plane-parallel potential motions of a heavy liquid of infinite depth which are generated by the action of a source with constant intensity, which is placed under the surface of the liquid. The surface of the liquid being horizontal in the absence of an acting source, will become wavy in the presence of the source. We have in mind to determine the form of these waves, not assuming that they are infinitely small, but still supposing that they are sufficiently small.

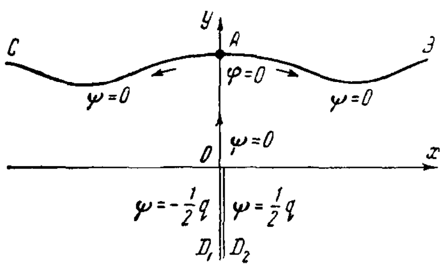

Fig. 1

We shall assume that the motion of the liquid takes place in the vertical plane x, O, y ; the origin of the coordinates will be taken at the source, the y-axis will be directed vertically upward; let A designate the point on the liquid surface located on the y-axis, and let B and C designate points on the liquid surface which have an infinite distance to A . Let us cut the flow plane along the y-axis from the point O to the point $y = -\infty$ and let us designate by D_1 and D_2 points which are located at infinity on two sides of the cut (Fig.1).

The flow line which originates at point O and which is directed along the y-axis from the source to point A , is symmetrically divided at this point into two sections, AB and AC . Now assign to this flow line a zero value as a function of the flow ψ and assume also that at the point A the velocity potential φ is equal to zero. The line of the flow which is direcred vertically downward from point O along the cut has two values in its flow function; along the left-hand side of the cut OD_1 we have $\psi = -\frac{1}{2}q$, and along the right-hand side of the cut OD_2 we have $\psi = \frac{1}{2}q$. Here q designates the rate of discharge of the source.

Let us map the entire flow region $BACD_1OD_2B$ located on the plane of the complex variable $z = x + iy$ onto the plane $w = \varphi + i\psi$. In this latter plane we will have a strip of width q, symmetrically located with rspect to the φ-axis, and possessing a cut BAC which passes along the negative part of this axis (Fig.2). The corresponding points on the planes z and w will be designated by the same letters.

The function which establishes the correspondence between Figs. 1 and 2 is, of course, unknown. Therefore the problem at hand will consist in determining this function.

Let us map the plane of the complex variable w into the plane of the auxiliary variable ζ, assuming that

$$\frac{dw}{d\zeta} = -\frac{q}{2\pi}\tan\frac{1}{2}\zeta \tag{1.1}$$

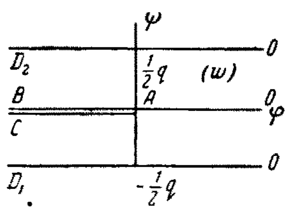

Fig. 2

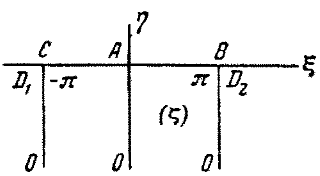

Fig. 3

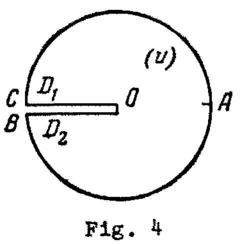

Fig. 4

On the plane of the variable ζ we will have a vertical semistrip as shown in Fig.3. Finally, we transform this semistrip on the plane of the primary auxiliary variable u, assuming that $u = e^{-i\zeta}$.

In the plane of the variable u we will have, as the region common to the entire flow along the z-plane, a circle with a radius equal to unity and a cut along the straight line $(u = -1,\ u = 0)$ (Fig.4). The corresponding points of the planes z, w, u are designated by the same letters. In passing from the variables ζ to the variable u, Formula (1.1) is replaced by the following:

$$\frac{dw}{du} = -\frac{q}{2\pi u}\frac{1-u}{1+u} \tag{1.2}$$

Now let us establish the connection between the complex variables z and u.

If on the plane z there was no source, then the correspondence between planes z and ζ would be expressed by Formula

$$\frac{dz}{d\zeta} = \frac{l}{8\pi}\frac{1}{\cos^2 {}^1/_2\zeta}$$

because in the absence of the source the line BAC would be a horizontal line; l is a certain constant related to the distance OA. When changing to u, the last formula becomes

$$\frac{dz}{du} = \frac{il}{2\pi}\frac{1}{(u+1)^2}$$

Assume that in the presence of the source the connection between the planes z and u is expressed by Formula

$$\frac{dz}{du} = \frac{il}{2\pi}\frac{1}{(u+1)^2}\frac{1}{f(u)} \tag{1.3}$$

where $f(u)$ is an unknown function, holomorphic inside the circle $|u| = 1$. Let us expand this function following the MacLaurin's theorem as follows:

$$f(u) = 1 + b_1 u + b_2 u^2 + \ldots \tag{1.4}$$

The problem at hand is to determine this function.

2. In order to find the function $f(u)$ let us set up the equation which would transfer the constant pressure condition along the free surface BAC for the fluid. On the basis of the Bernoulli integral this condition will be expressed

$$V^2 + 2gy = \text{const} \tag{2.1}$$

Here V is the particle velocity on the free fluid surface at point having an ordinate y.

According to Formulas of the preceding section we will determine the quantities V and y. We have

$$\frac{dw}{dz} = \frac{iq}{l}\frac{1-u^2}{u} f(u)$$

Let us apply this formula to the points located on the fluid surface where $u = e^{i\theta}$, then we will have

$$\frac{dw}{dz} = \frac{iq}{l}(e^{-i\theta} - e^{i\theta}) f(e^{i\theta}), \qquad \overline{\frac{dw}{dz}} = -\frac{iq}{l}[(e^{i\theta} - e^{-i\theta})\, \overline{f(e^{i\theta})}$$

Multiply termwise these equations, we have

$$V^2 = 2q^2 l^{-2}(1 - \cos 2\theta) f(e^{i\theta})\, \overline{f(e^{i\theta})} \tag{2.2}$$

From Formula (1.3) we will have

$$\frac{dz}{d\theta} = -\frac{l}{4\pi}\frac{1}{1+\cos\theta}\frac{1}{f(e^{i\theta})}, \qquad \frac{\overline{dz}}{d\theta} = -\frac{l}{4\pi}\frac{1}{1+\cos\theta}\frac{1}{\overline{f(e^{i\theta})}} \tag{2.3}$$

We then have

$$\frac{dx}{d\theta} = -\frac{l}{8\pi}\frac{1}{1+\cos\theta}\left\{\frac{1}{f(e^{i\theta})} + \frac{1}{\overline{f(e^{i\theta})}}\right\}, \quad \frac{dy}{d\theta} = -\frac{l}{8\pi i}\frac{1}{1+\cos\theta}\left\{\frac{1}{f(e^{i\theta})} - \frac{1}{\overline{f(e^{i\theta})}}\right\}$$

From Formulas (2.3) it is easy to see that the symmetrical property of the fluid surface relative to the OY-axis has as a consequence the following property of the function $f(u)$:

$$\overline{f(e^{i\theta})} = f(e^{-i\theta})$$

From here it follows that all the coefficients in the expansion (1.4) are indeed real numbers. Now we can rewrite Formulas (2.2) and (2.3) so that

$$V^2 = \frac{2q^2}{l^2}(1 - \cos 2\theta) f(e^{i\theta}) f(e^{-i\theta}), \quad \frac{dy}{d\theta} = -\frac{l}{8\pi i}\frac{1}{1+\cos\theta}\left\{\frac{1}{f(e^{i\theta})} - \frac{1}{f(e^{-i\theta})}\right\}$$

Differentiate now the basic equation (2.1) with respect to the variable θ and write the result using the preceding formulas. We will then have

$$\varepsilon \frac{d}{d\theta}[(1 - \cos 2\theta) f(e^{i\theta}) f(e^{-i\theta})] - \frac{1}{1+\cos\theta}\frac{1}{2i}\left\{\frac{1}{f(e^{i\theta})} - \frac{1}{f(e^{-i\theta})}\right\} = 0$$

$$\left(\varepsilon = \frac{4\pi q^2}{gl^3}\right) \tag{2.4}$$

This is the basic equation for the problem under consideration; taking advantage of this equation we can find the coefficient of the expansion (1.4).

3. Proceeding to solve Equation (2.4), first let us indicate certain auxiliary formulas. We have

$$f(e^{i\theta}) = 1 + b_1 e^{i\theta} + b_2 e^{2i\theta} + b_3 e^{3i\theta} + \ldots, \qquad f(e^{-i\theta}) = 1 + b_1 e^{-i\theta} + b_2 e^{-2i\theta} + b_3 e^{-3i\theta} + \ldots$$

From here we obtain

$$f(e^{i\theta}) f(e^{-i\theta}) = {}^1/_2 B_0 + B_1 \cos\theta + B_2 \cos 2\theta + B_3 \cos 3\theta + \ldots$$

where $B_0, B_1, B_2, B_3, \ldots$ are defined with Formulas

$$\begin{aligned} {}^1/_2 B_0 &= 1 + b_1^2 + b_2^2 + b_3^2 + b_4^2 + \ldots & {}^1/_2 B_2 &= b_2 + b_1 b_3 + b_2 b_4 + b_3 b_5 + \ldots \\ {}^1/_2 B_1 &= b_1 + b_1 b_2 + b_2 b_3 + b_3 b_4 + \ldots & {}^1/_2 B_3 &= b_3 + b_1 b_4 + b_2 b_5 + b_3 b_6 + \ldots \\ {}^1/_2 B_4 &= b_4 + b_1 b_5 + b_2 b_6 + b_3 b_7 + \ldots & &\text{and so forth} \end{aligned} \qquad (3.1)$$

Further we have

$$\begin{aligned} &\frac{d}{d\theta}[(1-\cos 2\theta) f(e^{i\theta}) f(e^{-i\theta})] = -(B_1 - {}^1/_2 B_1 - {}^1/_2 B_3)\sin\theta + \\ &- 2(B_2 - {}^1/_2 B_0 - {}^1/_2 B_4)\sin 2\theta - 3(B_3 - {}^1/_2 B_1 - {}^1/_2 B_5)\sin 3\theta - \\ &- 4(B_4 - {}^1/_2 B_2 - {}^1/_2 B_6)\sin 4\theta - 5(B_5 - {}^1/_2 B_3 - {}^1/_2 B_7)\sin 5\theta - \ldots \end{aligned}$$

Using this formula we obtain

$$\begin{aligned} &(1+\cos\theta)\frac{d}{d\theta}[(1-\cos 2\theta) f(e^{i\theta}) f(e^{-i\theta})] = \\ &= ({}^1/_2 B_1 + {}^1/_2 B_0 - B_1 - B_2 + {}^1/_2 B_3 + {}^1/_2 B_4)\sin\theta + \\ &+ ({}^1/_4 B_1 + B_0 + {}^1/_4 B_1 - 2B_2 - {}^5/_4 B_3 + B_4 + {}^3/_4 B_5)\sin 2\theta + \\ &+ ({}^1/_2 B_0 + {}^3/_2 B_1 - 3B_3 - {}^3/_2 B_4 + {}^3/_2 B_5 + B_6)\sin 3\theta + \\ &+ ({}^3/_4 B_1 + 2B_2 - {}^1/_4 B_3 - 4B_4 - {}^7/_4 B_5 + 2B_6 + {}^5/_4 B_7)\sin 4\theta + \\ &+ (B_2 + {}^5/_2 B_3 - {}^1/_2 B_4 + 5B_5 - 2B_6 + {}^5/_2 B_7 + {}^3/_2 B_8)\sin 5\theta + \ldots \end{aligned} \qquad (3.2)$$

Then we set

$$1 / f(u) = 1 + c_1 u + c_2 u^2 + c_3 u^3 + \ldots \qquad (3.3)$$

Coefficients of this expansion are connected with the coefficients of Equation (1.4), as follows:

$$\begin{aligned} &b_1 + c_1 = 0, \quad b_3 + c_1 b_2 + c_2 b_1 + c_3 = 0, \qquad b_4 + c_1 b_3 + c_2 b_2 + c_3 b_1 + c_4 = 0 \\ &b_2 + c_1 b_1 + c_2 = 0, \qquad b_5 + c_1 b_4 + c_2 b_3 + c_3 b_2 + c_4 b_1 + c_5 = 0 \quad \text{etc.} \end{aligned} \qquad (3.4)$$

We have

$$\frac{1}{2i}\left[\frac{1}{f(e^{i\theta})} - \frac{1}{f(e^{-i\theta})}\right] = c_1 \sin\theta + c_2 \sin 2\theta + c_3 \sin 3\theta + \ldots \qquad (3.5)$$

4. We substitute the expansions (3.2) and (3.5) into Equation (2.4). Equating the coefficients of the sines of different multiples of a given arc, we will have

$$\begin{aligned} c_1 &= \varepsilon\,[{}^1/_2 B_1 + {}^1/_2 B_0 - B_1 - B_2 + {}^1/_2 B_3 + {}^1/_2 B_4] \\ c_2 &= \varepsilon\,[{}^1/_4 B_1 + B_0 + {}^1/_4 B_1 - 2B_2 - {}^5/_4 B_3 + B_4 + {}^3/_4 B_5] \\ c_3 &= \varepsilon\,[{}^1/_2 B_0 + {}^3/_2 B_1 - 3B_3 - {}^3/_2 B_4 + {}^3/_2 B_5 + B_6] \\ c_4 &= \varepsilon\,[{}^3/_4 B_1 + 2B_2 - {}^1/_4 B_3 - 4B_4 - {}^7/_4 B_5 + 2B_6 + {}^5/_4 B_7] \\ c_5 &= \varepsilon\,[B_2 + {}^5/_2 B_3 - {}^1/_2 B_4 - 5B_5 - 2B_6 + {}^5/_2 B_7 + {}^3/_2 B_8] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \qquad (4.1)$$

Supplementing this system by Equations (3.4), we can find the unknown coefficients $b_1, b_2, b_3, b_4, \ldots, c_1, c_2, c_3, c_4, \ldots$ We will look for these coefficients in terms of series

$$b_1 = \varepsilon(b_{10} + b_{11}\varepsilon + b_{12}\varepsilon^2 + \ldots), \qquad b_2 = \varepsilon(b_{20} + b_{21}\varepsilon + b_{22}\varepsilon^2 + \ldots)$$
$$b_3 = \varepsilon(b_{30} + b_{31}\varepsilon + b_{32}\varepsilon^2 + \ldots), \qquad b_4 = \varepsilon(b_{40} + b_{41}\varepsilon + b_{42}\varepsilon^2 + \ldots) \quad \text{and so forth}$$

$$c_1 = \varepsilon(c_{10} + c_{11}\varepsilon + c_{12}\varepsilon^2 + \ldots), \qquad c_2 = \varepsilon(c_{20} + c_{21}\varepsilon + c_{22}\varepsilon^2 + \ldots)$$
$$c_3 = \varepsilon(c_{30} + c_{31}\varepsilon + c_{32}\varepsilon^2 + \ldots), \qquad c_4 = \varepsilon(c_{40} + c_{41}\varepsilon + c_{42}\varepsilon^2 + \ldots) \quad \text{and so forth}$$

The substitution of these series into Equations (3.4) and (4.1) allows us to find their unknown coefficients. By equating these coefficients of various powers of ε on both sides of the given equations, we have the following results:

$$\begin{array}{lllllll}
c_{10} = 1, & c_{11} = 4, & c_{50} = 0, & c_{71} = 0, & c_{12} = 32, & c_{52} = -169, & c_{92} = 14 \\
c_{20} = 2, & c_{21} = {}^{19}/_2, & c_{60} = 0, & c_{81} = 0, & c_{22} = 79, & c_{62} = 31, & c_{102} = 0 \\
c_{30} = 1, & c_{31} = 3, & c_{51} = -9, & c_{91} = 0, & c_{32} = -4, & c_{72} = 129 & \\
c_{40} = 0, & c_{41} = -9, & c_{61} = -{}^{5}/_2, & & c_{42} = -182, & c_{82} = {}^{147}/_2 &
\end{array}$$

Thus, we can sequentially write the expansion of the function $1/f(u)$ according to the powers of u, accounting for all the terms up to the third order of smallness of the parameteric value ε

$$\begin{aligned}
1/f(u) = 1 &+ \varepsilon(1 + 4\varepsilon + 32\varepsilon^2 + \ldots)\, u + \varepsilon(2 + {}^{19}/_2\varepsilon + 79\varepsilon^2 + \ldots)\, u^2 + \\
&+ \varepsilon(1 + 3\varepsilon - 4\varepsilon^2 + \ldots)\, u^3 + \varepsilon(-9\varepsilon - 182\varepsilon^2 + \ldots)\, u^4 + \\
&+ \varepsilon(-9\varepsilon - 169\varepsilon^2 + \ldots)\, u^5 + \varepsilon(-{}^{5}/_2\varepsilon + 31\varepsilon^2 + \ldots)\, u^6 + \\
&+ \varepsilon(129\varepsilon^2 + \ldots)\, u^7 + \varepsilon({}^{147}/_2\varepsilon^2 + \ldots)\, u^8 + \varepsilon(14\varepsilon^2 + \ldots)\, u^9 + \ldots
\end{aligned} \tag{4.2}$$

5. Substitution of series (4.2) into Formula (1.3) gives the following result:

$$\begin{aligned}
\frac{2\pi}{il}\frac{dz}{du} &= \frac{1 - {}^{1}/_2\varepsilon^2}{(u+1)^2} + \frac{1}{2}\varepsilon^3 + [(\varepsilon + 4\varepsilon^2 + 31\varepsilon^3)\, u + ({}^{3}/_2\varepsilon^2 + {}^{33}/_2\varepsilon^3)\, u^2 + \\
&+ (-4\varepsilon^2 - 68\varepsilon^3)\, u^3 + (-{}^{5}/_2\varepsilon^2 - {}^{125}/_2\varepsilon^3)\, u^4 + 24\varepsilon^3u^5 + {}^{91}/_2\varepsilon^3u^6 + 14\varepsilon^3u^7]
\end{aligned}$$

From here we have with accuracy to the third order in ε

$$\begin{aligned}
z = \frac{il}{2\pi}\int_0^u \frac{1}{(u+1)^2}\frac{du}{f(u)} &= \frac{il}{2\pi}\Big[-\frac{1 - {}^{1}/_2\varepsilon^2}{1+u} + (1 - {}^{1}/_2\varepsilon^2) + {}^{1}/_2\varepsilon^3u + \\
&+ {}^{1}/_2(\varepsilon + 4\varepsilon^2 + 31\varepsilon^3)\, u^2 + {}^{1}/_2(\varepsilon^2 + 11\varepsilon^3)\, u^3 - (\varepsilon^2 + 17\varepsilon^3)\, u^4 - \\
&- {}^{1}/_2(\varepsilon^2 + 25\varepsilon^3)\, u^5 + 4\varepsilon^3u^6 + {}^{13}/_2\varepsilon^3u^7 + {}^{7}/_4\varepsilon^3u^8 + \ldots\Big]
\end{aligned}$$

By assuming $u = e^{i\theta}$ and separating the imaginary part from the real one, we find a parameteric equation for the surface of the liquid.

$$\begin{aligned}
2\pi x/l = &-{}^{1}/_2(1 - {}^{1}/_2\varepsilon^2)\tan {}^{1}/_2\theta - {}^{1}/_2\varepsilon^3\sin\theta - {}^{1}/_2(\varepsilon + 4\varepsilon^2 + 31\varepsilon^3)\sin 2\theta - \\
&- {}^{1}/_2(\varepsilon^2 + 11\varepsilon^3)\sin 3\theta + (\varepsilon^2 + 17\varepsilon^3)\sin 4\theta + {}^{1}/_2(\varepsilon^2 + 25\varepsilon^3)\sin 5\theta - 4\varepsilon^3\sin 6\theta - \\
&- {}^{13}/_2\varepsilon^3\sin 7\theta - {}^{7}/_4\varepsilon^3\sin 8\theta + \ldots
\end{aligned}$$

$$\begin{aligned}
2\pi y/l = &\,{}^{1}/_2(1 - {}^{1}/_2\varepsilon^2) + {}^{1}/_2\varepsilon^3\cos\theta + {}^{1}/_2(\varepsilon + 4\varepsilon^2 + 31\varepsilon^3)\cos 2\theta + {}^{1}/_2(\varepsilon^2 + 11\varepsilon^3)\cos 3\theta - \\
&- (\varepsilon^2 + 17\varepsilon^3)\cos 4\theta - {}^{1}/_2(\varepsilon^2 + 25\varepsilon^3)\cos 5\theta + 4\varepsilon^3\cos 6\theta + {}^{13}/_2\varepsilon^3\cos 7\theta + {}^{7}/_4\varepsilon^3\cos 8\theta + \ldots
\end{aligned}$$

We set in the second equation $\theta = \pi$; for this value of θ the ordinate y will be equal to the height of the liquid surface above its level at infinity, or, as can be also said, the depth h of the submerged source, we have

$$2\pi h/l = {}^{1}/_2(1 + \varepsilon + {}^{3}/_2\varepsilon^2 + {}^{17}/_2\varepsilon^3 + \ldots)$$

From this formula with the aid of a specified value for h, we find the auxiliary parameter l which is contained in the surface equation.

By assuming that $\theta = 0$, we can find the ordinate for the surface above the source. Calculations show that this ordinate will be equal to h, as it should be.

6. We return to Eqiation (2.4) and transform it by introducing instead of function $f(u)$ the function

$$\omega = \tau + i\vartheta$$

where ϑ is the angle of inclination of the velocity to the OX-axis and τ is defined as

$$\tau = \ln\left[\left(\frac{gq}{2\pi}\right)^{1/3}\left|\frac{dz}{dw}\right|\right]$$

In carrying out appropriate calculations, we can find the relation between τ and ϑ on the circumference $|u| = 1$

$$\frac{d\tau}{d\theta} - \tan\frac{1}{2}\theta\, e^{3\tau}\sin\vartheta = 0 \qquad (6.1)$$

This condition resembles the Levi-Civita condition, but it does not include any type of parameter. According to the function ω the function $f(e^{i\theta})$ is determined as

$$f(e^{i\theta}) = -\frac{l}{2q}\left(\frac{gq}{2\pi}\right)^{1/3}\frac{e^{-\omega}}{\sin\theta}$$

This function will be a particular form of the general formula

$$f(u) = \frac{il}{q}\left(\frac{gq}{2\pi}\right)^{1/3}\frac{u}{1-u^2}e^{-\omega} \qquad (6.2)$$

The function $\omega(u)$ can be determined by using the boundary condition (6.1) in the form of a series of powers of parameter ϵ introduced above and then by using Formula (6.2) we can again arrive at series (4.2).

Translated by V.M.G.